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Bernstein–Sato polynomials in positive characteristic

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ABSTRACT

In characteristic zero, the Bernstein–Sato polynomial of a hypersurface can be described as the minimal polynomial of the action of an Euler operator on a suitable D -module. We consider analogous D -modules in positive characteristic, and use them to define a sequence of Bernstein–Sato polynomials (corresponding to the fact that we need to consider also divided powers Euler operators). We show that the information contained in these polynomials is equivalent to that given by the F -jumping exponents of the hypersurface, in the sense of Hara and Yoshida [N. Hara, K.-i. Yoshida, A generalization of tight closure and multiplier ideals, *Trans. Amer. Math. Soc.* 355 (2003) 3143–3174].

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1. Introduction

The goal of this note is to describe a connection between the theory of generalized test ideals, in the sense of Hara and Yoshida [HY], and the theory of D -modules. Suppose that $X = \text{Spec}(R)$ is a nonsingular affine scheme, and that $f \in R$ is a nonzero regular function on X .

Let us describe first the characteristic zero situation, studied by Malgrange in [Mal]. If $\iota: X \rightarrow X \times \mathbf{A}^1$ is the graph of f , let $B_f := \iota_+ \mathcal{O}_X$ denote the D -module theoretic push-forward of the structure sheaf of X . This has the following explicit description as the first local cohomology module of $X \times \mathbf{A}^1$ along the image of ι

$$B_f \simeq H_{\iota(X)}^1(\mathcal{O}_{X \times \mathbf{A}^1}) \simeq R[t]_{f-t}/R[t]. \quad (1)$$

The class of $1/(f-t)$ in B_f is denoted by δ (this is the δ -function corresponding to the graph of f). Let D_R denote the ring of differential operators on R . Malgrange constructed the V -filtration on B_f ,

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which is a filtration by D_R -modules such that, informally speaking, $\partial_t t$ is put in upper-triangular form when passing to the graded module associated to this filtration. The key ingredient in this construction is the Bernstein–Sato polynomial $b_f(s) \in \mathbf{Q}[s]$, that in this context can be interpreted as the minimal polynomial of $-\partial_t t$ acting on the D_R -module

$$M_f/tM_f, \quad \text{where } M_f := D_R[\partial_t t] \cdot \delta. \quad (2)$$

We mention that a recent result of Budur and Saito [BS] relates the V -filtration to the theory of multiplier ideals as follows. Recall that for every nonnegative λ one defines the multiplier ideal $\mathcal{J}(f^\lambda) \subseteq \mathcal{O}_X$, and one gets in this way a decreasing filtration of \mathcal{O}_X (see Chapter 9 in [Laz]). If one considers the embedding $R \hookrightarrow B_f$ given by $h \mapsto h\delta$, then the V -filtration induces (up to a minor renormalization) the filtration on R given by the multiplier ideals of f .

Suppose now that $\text{char}(R) = p > 0$, and let us assume that R is F -finite, that is, the Frobenius morphism on R is finite. In this case, the ring of differential operators D_R is not finitely generated over R , but it can be written as a union of subrings D_R^e , where $D_R^e = \text{End}_{R^{p^e}}(R)$.

Our main point is that one can define the D_R -modules B_f and M_f also in the positive characteristic setting, and these D_R -modules are related to the generalized test ideals $\tau(f^\lambda)$ of Hara and Yoshida [HY]. As in the case of multiplier ideals, λ is a nonnegative real parameter. The generalized test ideals give a decreasing filtration of R , and the exponents where the test ideals change value are the F -jumping exponents of f . It was shown in [BMSm1] that the F -jumping exponents of f form a discrete set of rational numbers. We stress that unlike the multiplier ideals that are defined via a resolution of singularities, the test ideals are defined using the action of the Frobenius morphism on the ring. On the other hand, there are interesting results and conjectures relating the multiplier ideals and the test ideals via reduction mod p (see Section 3).

Note that in characteristic $p > 0$ we have an infinite set of Euler operators $\partial_{p^i} := \partial_t^{[p^i]} t^{p^i}$, for $i \geq 0$ (recall that $\partial_t^{[m]}$ is the differential operator whose action is given by $\partial_t^{[m]} \bullet t^r = \binom{r}{m} t^{r-m}$). Unlike in characteristic zero, the action of these operators on D -modules is easy to describe. In fact, every $D_{R[t]}^e$ -module admits a decomposition into common eigenspaces for the operators $\partial_1, \partial_p, \dots, \partial_{p^{e-1}}$ (the eigenvalues being in \mathbf{F}_p). In the case of the module B_f , we write down an R -basis of B_f consisting of common eigenvectors. Moreover, the action of D_R , t and $\partial_t^{[p^i]}$ on this basis can be described explicitly (see Theorem 5.5 for the precise statement).

Instead of only considering the D_R -module M_f/tM_f , in this case it is natural to consider separately all modules

$$M_f^e/tM_f^e, \quad \text{where } M_f^e = D_R^e[\partial_1, \dots, \partial_{p^{e-1}}] \cdot \delta. \quad (3)$$

The corresponding eigenspace decomposition for B_f induces a decomposition of M_f^e/tM_f^e into common eigenspaces for $\partial_1, \partial_p, \dots, \partial_{p^{e-1}}$, each eigenvalue lying in \mathbf{F}_p .

By analogy with the characteristic zero situation, we define the Bernstein–Sato polynomial of f to be the minimal polynomial of $-\partial_1$ acting on M_f^1/tM_f^1 . This is a product of linear forms in $\mathbf{F}_p[s]$, each appearing with multiplicity one. Note that if f is the reduction mod $p \gg 0$ of a polynomial $\tilde{f} \in \mathbf{Z}[x_1, \dots, x_n]$, then b_f divides the reduction mod p of $b_{\tilde{f}}$ (since $b_{\tilde{f}} \in \mathbf{Q}[x]$, the reduction mod p of $b_{\tilde{f}}$ makes sense if p is large enough).

In order to also keep track of the higher Euler operators it is more convenient to consider Bernstein–Sato polynomials with coefficients in \mathbf{Q} . We put $b_f^{(1)}(s) := \prod_i (s - \frac{i}{p})$, where the product is over those $i \in \{0, 1, \dots, p-1\}$ such that there is a nonzero eigenvector of $-\partial_1$ in M_f^1/tM_f^1 with eigenvalue $\bar{i} \in \mathbf{F}_p$. More generally, for every $e \geq 1$ we put

$$b_f^{(e)}(s) := \prod_{i_1, \dots, i_e} \left(s - \left(\frac{i_e}{p} + \dots + \frac{i_1}{p^e} \right) \right),$$

where the product is over those $i_1, \dots, i_e \in \{0, 1, \dots, p-1\}$ such that there is a nonzero $w \in M_f^e/tM_f^e$ with $(\partial_{p^{\ell-1}} + i_\ell)w = 0$ for every $1 \leq \ell \leq e$. In other words, the Bernstein–Sato polynomial $b_f^{(e)}$ describes the common eigenvalues of the operators $\partial_1, \partial_p, \dots, \partial_{p^{e-1}}$ acting on M_f^e/tM_f^e .

Our main result says that the information given by the polynomials $b_f^{(e)}$ is equivalent to that of the F -jumping exponents of f . If $\lambda > 1$, then λ is an F -jumping exponent if and only if $\lambda - 1$ has this property, and therefore it is enough to understand the F -jumping exponents in the interval $(0, 1]$ (recall that this is a finite set of rational numbers). In the next theorem, we denote by $\lceil u \rceil$ the smallest integer $\geq u$.

Theorem. *Let R be a regular F -finite ring of positive characteristic p . Consider the F -jumping exponents $\lambda_1, \dots, \lambda_r$ of f that lie in $(0, 1]$. Given $e \geq 1$, the rational number $\frac{\lceil p^e \lambda_i \rceil - 1}{p^e}$ is a root of the Bernstein–Sato polynomial $b_f^{(e)}$. Moreover, every root of $b_f^{(e)}$ is of this form, for some $i \in \{1, \dots, r\}$.*

We mention that the first connection between invariants in positive characteristic and Bernstein–Sato polynomials has been noticed in [MTW]. With the above notation, the result in [MTW] can be stated as follows. Suppose that \tilde{f} is defined over \mathbf{Z} , and that f is the reduction mod p of \tilde{f} , for some $p \gg 0$. If $b_{\tilde{f}}$ is the Bernstein–Sato polynomial of \tilde{f} , and if λ is an F -jumping exponent of f , then $\lceil p^e \lambda \rceil - 1$ is a root of $b_{\tilde{f}}$ mod p . The above theorem is a first step towards a better understanding of this connection between Bernstein–Sato polynomials and the generalized test ideals.

The paper is structured as follows. The first two sections are of an expository nature, reviewing the necessary notions from zero and positive characteristic. In Section 2 we give an introduction to the circle of ideas around the V -filtration. In particular, we describe the role of the Bernstein–Sato polynomial in this setting. In Section 3 we overview the definition of the generalized test ideals following [BMSm2]. We also discuss the most interesting results and conjectures about these ideals, concerning their connection with multiplier ideals via reduction mod p . In Section 4 we show that every $D_{R[t]}^e$ -module has a canonical decomposition into common eigenspaces with respect to the action of the Euler operators $\partial_1, \partial_p, \dots, \partial_{p^{e-1}}$. In Section 5 we turn to the case of the module B_f , and we write down an explicit basis of common eigenvectors. In the last section we define the Bernstein–Sato polynomials and prove the above theorem. We end with some examples and questions related to this setup.

2. Bernstein–Sato polynomials and V -filtrations

We recall in this section, following [Mal], the notion of V -filtration and its connection with the Bernstein–Sato polynomial. We work over a fixed algebraically closed field k of characteristic zero. For simplicity, we restrict to the hypersurface case, though a similar picture is known to hold for ideals of arbitrary codimension (see [BMSa]).

Let X be a smooth, connected n -dimensional variety, and let H be a hypersurface in X . Our invariants are local, hence we may and will assume that $X = \text{Spec}(R)$ is affine and H is defined by $(f = 0)$ for some nonzero $f \in R$. We denote by D_R the ring of differential operators on X (over k), and denote by $P \bullet h$ the action of $P \in D_R$ on $h \in R$. Around every point in X we can find a principal affine open subset $U = \text{Spec}(R_a)$ such that we have $x_1, \dots, x_n \in R_a$ that give an étale morphism $U \rightarrow \mathbf{A}^n$. If $\partial_1, \dots, \partial_n \in \text{Der}_k(R_a)$ are the corresponding derivations, then $D_{R_a} \simeq (D_R)_a$ is generated by R_a and $\partial_1, \dots, \partial_n$.

We now give the definition of the Bernstein–Sato polynomial. Consider an extra variable s , and the free $R_f[s]$ -module generated by the symbol f^s . This is, in fact, a left module over $D_{R_f}[s]$ if we let a derivation D of R_f act by

$$D \cdot f^s = \frac{sD(f)}{f} f^s.$$

It was shown by Bernstein that there is a nonzero $b(s) \in k[s]$ and $P \in D_R[s]$ (that is, P is a polynomial in s with coefficients in D_R) such that

$$b(s)f^s = P \cdot f^{s+1}. \quad (4)$$

It is clear that the set of polynomials $b(s)$ for which there is P satisfying (4) is an ideal in $k[s]$. The monic generator of this ideal is called the *Bernstein–Sato polynomial* of f , and it is denoted by b_f .

In (4) we have treated f^s as a formal symbol. However, this equation has the obvious meaning whenever we can make sense of f^s . For example, if $m \in \mathbb{Z}$, we can let $s = m$ in (4) and then we get a corresponding equality in R_f .

The Bernstein–Sato polynomial is a subtle invariant of the singularities of the hypersurface $H = (f = 0)$. A deep theorem of Kashiwara [Kas2] says that all roots of b_f are negative rational numbers. In particular, b_f has rational coefficients. One of the main applications of the V -filtration in [Mal] was to relate, when H has isolated singularities, the roots of b_f with the eigenvalues of the monodromy action on the Milnor fiber.

We now explain the definition of the V -filtration of f , and the connection with the Bernstein–Sato polynomial. Let $\iota: X \hookrightarrow X \times \mathbb{A}^1$ be the graph map of f , that is $\iota(x) = (x, f(x))$. We have a left D -module on $X \times \mathbb{A}^1$ (that is, a left $D_{R[t]}$ -module) given as the D -module push-forward of R , namely $B_f := \iota_+ R$. This can be explicitly described as the first cohomology module of $X \times \mathbb{A}^1$ along the graph of f

$$B_f \simeq R[t]_{f-t}/R[t].$$

Via this identification, the action of the differential operators on B_f is induced by the natural action on the localization of $R[t]$. It is easy to see that if we denote by δ the class of $\frac{1}{f-t}$ in B_f , then B_f is free over R with a basis given by

$$\partial_t^m \cdot \delta = \frac{m!}{(f-t)^{m+1}},$$

for $m \geq 0$.

Consider now the D_R -module $M_f := D_R[\partial_t] \cdot \delta \subset B_f$. One can show that

$$tM_f = D_R[\partial_t] \cdot t\delta = D_R[\partial_t] \cdot f\delta \subseteq M_f.$$

A key observation is that (4) holds if and only if

$$b(-\partial_t) \cdot \delta = P(-\partial_t)f \cdot \delta. \quad (5)$$

Indeed, consider the ring homomorphism $\varphi: D_{R_f}[s] \rightarrow D_{R_f}[\partial_t]$ given by $\varphi(P(s)) = P(-\partial_t)$. This makes $B_f \otimes_R R_f$ into a $D_{R_f}[s]$ -module. We also have a $D_{R_f}[s]$ -linear map $\psi: R_f[s]f^s \rightarrow B_f \otimes_R R_f$ given by $\psi(Q(s)f^s) = Q(-\partial_t) \cdot \delta$. To see that ψ is indeed linear with respect to the action of differential operators, note that if D is a derivation on R_f , then

$$\psi(D \cdot f^s) = \psi\left(\frac{sD(f)}{f} f^s\right) = -\partial_t \cdot \frac{D(f)}{f} \delta = -D(f)\partial_t \cdot \delta = D \cdot \delta.$$

Since $\{(\partial_t)^m \cdot \delta\}_{m \geq 0}$ gives a basis of $B_f \otimes_R R_f$ over R_f , it follows that ψ is injective. Using also the fact that $B_f \subseteq B_f \otimes_R R_f$, we deduce that (4) is equivalent with (5).

Moreover, it is easy to see that $b(-\partial_t) \cdot \delta \in tM_f$ if and only if $b(-\partial_t) \cdot M_f \subseteq tM_f$. We conclude that b_f is the minimal polynomial of the action of $-\partial_t$ on M_f/tM_f .

The V -filtration is a decreasing filtration on B_f by finitely generated left D_R -submodules $\{V^\alpha\}_{\alpha \in \mathbb{Q}}$, with the following properties:

- (i) $\bigcup_{\alpha \in \mathbf{Q}} V^\alpha = B_f$.
- (ii) The filtration is semicontinuous and discrete in the following sense: there is a positive integer ℓ such that for every integer m and every $\alpha \in (\frac{m-1}{\ell}, \frac{m}{\ell}]$ we have $V^\alpha = V^{m/\ell}$.
- (iii) We have $t \cdot V^\alpha \subseteq V^{\alpha+1}$ for every α , with equality if $\alpha > 0$.
- (iv) We have $\partial_t \cdot V^\alpha \subseteq V^{\alpha-1}$ for every α .
- (v) For every α , if we put $V^{>\alpha} := \bigcup_{\beta > \alpha} V^\beta$, then $(\partial_t - \alpha)$ is nilpotent on $V^\alpha / V^{>\alpha}$.

The key property is (v) above. One can consider the V -filtration as an attempt to put the operator ∂_t on B_f in upper triangular form. It is not hard to show that if a filtration as above exists, then it is unique. Malgrange proved the existence of such a filtration in [Mal], using only the existence of the Bernstein–Sato polynomial and the rationality of its roots.

There is, in fact, an explicit description of the V -filtration in terms of more general Bernstein–Sato polynomials, due to Sabbah [Sab]. One can show (for example, using the existence of the V -filtration) that for every $w \in B_f$ there is a nonzero polynomial $b(s) \in \mathbf{Q}[s]$ and $P \in D_R[s]$ such that

$$b(-\partial_t)t w = P(-\partial_t)t \cdot w. \quad (6)$$

The set of polynomials $b(s)$ for which there is P as above is an ideal, and its monic generator is called the Bernstein–Sato polynomial of f associated to w , and it is denoted by $b_{f,w}$. Note that we have $b_f = b_{f,\delta}$. It is a consequence of the existence of the V -filtration that all roots of $b_{f,w}$ are rational. Using this terminology, Sabbah showed that V^α is the subset of B_f consisting of those w such that all roots of $b_{f,w}$ are $\leq -\alpha$.

We end this section with a result of Budur and Saito relating the V -filtration to the multiplier ideals of f . Recall that given f , we can use a log resolution of singularities for the pair (X, H) to attach to every $\lambda \in \mathbf{R}_+$ an ideal in R called the *multiplier ideal* of f of exponent λ , and denoted by $\mathcal{J}(f^\lambda)$. We refer to [Laz], Chapter 9, for the precise definition and for the basic properties. If $\lambda > \mu$, then $\mathcal{J}(f^\lambda) \subseteq \mathcal{J}(f^\mu)$. Moreover, given $\lambda \in \mathbf{R}_+$, there is $\varepsilon > 0$ such that $\mathcal{J}(f^\lambda) = \mathcal{J}(f^{\lambda+\varepsilon})$. A *jumping exponent* of f is a positive λ such that $\mathcal{J}(f^\lambda) \subsetneq \mathcal{J}(f^{\lambda-\varepsilon})$ for every $\varepsilon > 0$. We make the convention that 0 is also a jumping exponent. It follows easily from definition that all jumping exponents are rational and that they form a discrete subset of \mathbf{R}_+ . Since we consider only principal ideals, we also have $\mathcal{J}(f^{\lambda+1}) = f \cdot \mathcal{J}(f^\lambda)$, hence λ is a jumping exponent if and only if $\lambda + 1$ is.

Note that we have an embedding $R \hookrightarrow B_f$ given by $h \mapsto h\delta$. Budur and Saito showed in [BS] that the restriction to R of the V -filtration is, essentially, the filtration of R by multiplier ideals. More precisely, they showed that for every $\lambda \in \mathbf{R}_+$ we have $\mathcal{J}(f^\lambda) = V^{\lambda+\varepsilon} \cap R$ for $0 < \varepsilon \ll 1$. One deduces as an easy consequence of their statement the following result from [ELSV]: if $\lambda \in (0, 1]$ is a jumping exponent of f , then $b_f(-\lambda) = 0$. Note also that in light of Sabbah's description of the V -filtration, the result of Budur and Saito can be reinterpreted as saying that for $h \in R$ we have

$$\sup\{\alpha \in \mathbf{R}_+ \mid h \in \mathcal{J}(f^\alpha)\} = -\max\{\beta \mid b_{f,h\delta}(\beta) = 0\}.$$

3. Generalized test ideals

Hara and Yoshida introduced in [HY] a characteristic p analogue of the multiplier ideals, the (generalized) test ideals. In this section we recall the definition of these ideals, and their connection with the multiplier ideals via reduction mod p . In fact, since our ambient variety is nonsingular, we find it more convenient to work with an equivalent definition from [BMSm2]. We stick to the hypersurface case, as in the rest of the paper, though for most results in this section the extension to the case of arbitrary ideals is verbatim.

We fix a regular domain R of positive characteristic p . We always assume R to be F -finite (that is, the Frobenius homomorphism $F: R \rightarrow R$ given by $F(u) = u^p$ is finite). Note that since R is regular, F is also flat, hence R is locally free over R^p . Basic examples are $k[x_1, \dots, x_n]$ or $k[[x_1, \dots, x_n]]$, where k is a perfect field (or more generally, such that $[k:k^p]$ is finite).

If J is an ideal in R and $e \geq 1$, we denote by $J^{[p^e]}$ the e th Frobenius power of J , that is, the ideal generated by the p^e -powers of the elements in J

$$J^{[p^e]} = (u^{p^e} \mid u \in J).$$

If \mathfrak{b} is an arbitrary ideal, then one can easily deduce from the fact that R is locally free over R^{p^e} that among the ideals J such that $\mathfrak{b} \subseteq J^{[p^e]}$ there is a unique minimal one, that we denote by $\mathfrak{b}^{[1/p^e]}$.

If R is free over R^p , then one can compute $\mathfrak{b}^{[1/p^e]}$ as follows. Since R is free also over R^{p^e} , we can choose a basis y_1, \dots, y_N of R over R^{p^e} . Consider generators h_1, \dots, h_r of \mathfrak{b} , and write for every i

$$h_i = \sum_{j=1}^N a_{i,j}^{p^e} y_j,$$

with $a_{i,j} \in R$. With this notation, we have $\mathfrak{b}^{[1/p^e]} = (a_{i,j} \mid i, j)$.

We now fix a nonzero $f \in R$ and a nonnegative real number λ . One can check using the definition that for every $e \geq 1$ we have

$$(f^{[\lambda p^e]})^{[1/p^e]} \subseteq (f^{[\lambda p^{e+1}]})^{[1/p^{e+1}]}.$$

Since R is Noetherian, it follows that for $e \gg 0$ the ideal $(f^{[\lambda p^e]})^{[1/p^e]}$ does not depend on e . This is the (generalized) test ideal $\tau(f^\lambda)$. It is easy to see that if $\lambda = m/p^e$ for a nonnegative integer m , then $\tau(f^\lambda) = (f^m)^{[1/p^e]}$ (see, for example, Lemma 2.1 in [BMSm1]).

Note that $\tau(f^0) = R$. It follows from definition that if $\lambda > \mu$, then $\tau(f^\lambda) \subseteq \tau(f^\mu)$. It is shown in [BMSm2] that for every nonnegative λ there is $\varepsilon > 0$ such that $\tau(f^\lambda) = \tau(f^{\lambda+\varepsilon})$. A positive λ is called an *F-jumping exponent* if $\tau(f^\lambda) \neq \tau(f^{\lambda-\varepsilon})$ for every $\varepsilon > 0$. We make the convention that 0 is also an *F-jumping exponent*.

It is again easy to see from definition that $\tau(f^{\lambda+1}) = f \cdot \tau(f^\lambda)$, hence λ is an *F-jumping exponent* if and only if $\lambda + 1$ is. Other properties are more subtle: it is shown in [BMSm1] that every *F-jumping exponent* is rational, and that the set of *F-jumping exponents* is discrete in \mathbf{R} (see also [KLZ]).

Remark 3.1. We mention an interpretation of the *F-jumping exponents* as *F-thresholds* (see Proposition 2.7 in [MTW] and Corollary 2.30 in [BMSm2]). Let J be an ideal in R such that $f \in \text{Rad}(J)$. For every $e \geq 1$, we denote by $\nu^J(p^e)$ the largest $r \in \mathbf{N}$ such that $f^r \notin J^{[p^e]}$ (if there is no such r , then we put $\nu^J(p^e) = 0$).

It is easy to see that we have $\sup_e \frac{\nu^J(p^e)}{p^e} = \lim_{e \rightarrow \infty} \frac{\nu^J(p^e)}{p^e} < \infty$, and this limit is called the *F-threshold* of f with respect to J , and denoted by $c^J(f)$. One shows that the set of *F-jumping exponents* of f is equal to the set $\{c^J(f) \mid f \in \text{Rad}(J)\}$.

We also note that one can show that if $J \neq R$ (in which case $c^J(f) > 0$), then $\nu^J(p^e) = \lceil c^J(f)p^e \rceil - 1$ (see Proposition 1.9 in [MTW]).

Arguably the most interesting questions in this area involve the connections between multiplier ideals and test ideals, via reduction mod p . We now state the fundamental result, due to Hara and Yoshida.

Suppose that R is a domain that is smooth over \mathbf{Z} (in particular, it is of finite type over \mathbf{Z}). Let $f \in R$ be nonzero. For every prime p and every ideal \mathfrak{a} in R , we denote by \mathfrak{a}_p the image of \mathfrak{a} in $R_p := R \otimes_{\mathbf{Z}} \mathbf{F}_p$, where $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. We take a log resolution of $(R \otimes_{\mathbf{Z}} \mathbf{Q}, f)$, and we choose $a \in \mathbf{Z}$ such that this resolution is defined over $R \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{a}]$. If $p \gg 0$, then we may reduce the resolution mod p , such that it gives a log resolution of (R_p, f_p) . In fact, since $p \gg 0$ we may also assume that the push-forward sheaves that come up in the construction of multiplier ideals commute with base-change over \mathbf{Z} (note that we essentially deal with finitely many ideals).

Theorem 3.2. (See [HY].) *With the above notation, we have the following:*

(i) *If $p \gg 0$, then for every $\lambda \in \mathbf{R}_+$ we have*

$$\tau(f_p^\lambda) \subseteq \mathcal{J}(f^\lambda)_p.$$

(ii) *Moreover, for every λ , if $p \gg 0$, then we have equality*

$$\tau(f_p^\lambda) = \mathcal{J}(f^\lambda)_p.$$

The first assertion in the above theorem is proved by interpreting both the test ideal and the multiplier ideal in terms of local cohomology. The second part is much more subtle, making use of the Frobenius action on the de Rham complex, and of vanishing theorems in positive characteristic, following Deligne and Illusie [DI].

Conjecture 3.3. *With the notation in Theorem 3.2, there are infinitely many primes p such that for all $\lambda \in \mathbf{R}_+$ we have*

$$\tau(f_p^\lambda) = \mathcal{J}(f^\lambda)_p.$$

To illustrate the above behavior, we give two examples.

Example 3.4. We first treat the case of the cusp $f = x^2 + y^3 \in \mathbf{Z}[x, y]$. Because of the periodicity properties of both multiplier ideals and test ideals, it is enough to only consider exponents in $[0, 1)$. It follows from the well-known computation of the multiplier ideals of the cusp in characteristic zero (see Example 9.2.15 in [Laz]) that if $p \gg 0$, then

$$\mathcal{J}(f^\lambda)_p = \begin{cases} \mathbf{F}_p[x, y], & \text{for } 0 \leq \lambda < \frac{5}{6}; \\ (x, y), & \text{for } \frac{5}{6} \leq \lambda < 1. \end{cases}$$

On the other hand, we claim that if $p > 3$, then

$$\tau(f_p^\lambda) = \begin{cases} \mathbf{F}_p[x, y], & \text{for } 0 \leq \lambda < c(f_p); \\ (x, y), & \text{for } c(f_p) \leq \lambda < 1, \end{cases}$$

where $c(f_p) = \frac{5}{6}$ if $p \equiv 1 \pmod{3}$, and $c(f_p) = \frac{5}{6} - \frac{1}{6p}$ if $p \equiv 2 \pmod{3}$.

Indeed, the fact that $\tau(f_p^\lambda) = \mathbf{F}_p[x, y]$ if and only if $\lambda < c(f_p)$ was shown in [MTW], Example 4.3. In order to complete the proof of the claim it is enough to show that $(x, y) \subseteq (f^{p^e-1})^{[1/p^e]}$ for every $e \geq 1$. Indeed, since the origin is the only singular point of f , it follows that if $\lambda < 1$, then $\tau(f^\lambda) \subseteq (x, y)$. On the other hand, if $\lambda < 1$ and e is large enough, then $\lambda p^e \leq p^e - 1$, hence

$$(x, y) \subseteq (f^{p^e-1})^{[1/p^e]} \subseteq (f^{\lceil \lambda p^e \rceil})^{[1/p^e]},$$

which implies that $\tau(f^\lambda) = (x, y)$.

Note first that for every $0 \leq a \leq p^e - 1$, the binomial coefficient $\binom{p^e-1}{a}$ is not zero in \mathbf{F}_p . Indeed, this follows from the fact that the order of p in $m!$ is $\sum_{i \geq 1} \lfloor m/p^i \rfloor$, and the fact that for every $1 \leq e' \leq e - 1$ we have

$$\lfloor a/p^{e'} \rfloor + \lfloor (p^e - 1 - a)/p^{e'} \rfloor - \lfloor (p^e - 1)/p^{e'} \rfloor = p^{e-e'} - \lceil (a+1)/p^{e'} \rceil + \lfloor a/p^{e'} \rfloor - (p^{e-e'} - 1) = 0.$$

We now compute $(f^{p^e-1})^{[1/p^e]}$ by writing f^{p^e-1} in the basis of $\mathbf{F}_p[x, y]$ over $\mathbf{F}_p[x^{p^e}, y^{p^e}]$ given by $\{x^i y^j \mid 0 \leq i, j \leq p^e - 1\}$. Since the monomial

$$(x^2)^{\frac{p^e-1}{2}} (y^3)^{\frac{p^e-1}{2}} = y^{p^e} \cdot x^{p^e-1} y^{\frac{p^e-3}{2}}$$

appears with a nonzero coefficient in f^{p^e-1} , we see that $y \in (f^{p^e-1})^{[1/p^e]}$. Since $(x^2)^{p^e-1} = x^{p^e} \cdot x^{p^e-2}$ appears with coefficient one in f^{p^e-1} , we deduce that $x \in (f^{p^e-1})^{[1/p^e]}$. This completes the proof of our claim. Note that Conjecture 3.3 is satisfied in this case.

Example 3.5. We consider the case of a homogeneous polynomial $f \in \mathbf{Z}[x_1, \dots, x_n]$ of degree d that defines a hypersurface Y in $\mathbf{P}_\mathbf{Z}^{n-1}$. We assume that Y is nonsingular over \mathbf{C} , that is, f has an isolated singular point at the origin. Let $p \gg 0$. The usual computation of multiplier ideals using the blowing up at the origin shows that if $d \leq n$, then $\mathcal{J}(f^\lambda)_p = \mathbf{F}_p[x_1, \dots, x_n]$ for every $\lambda < 1$, and if $d > n$, then

$$\mathcal{J}(f^\lambda)_p = \begin{cases} \mathbf{F}_p[x_1, \dots, x_n], & \text{for } 0 \leq \lambda < \frac{n}{d}; \\ (x_1, \dots, x_n), & \text{for } \frac{n}{d} \leq \lambda < \frac{n+1}{d}; \\ \vdots & \vdots \\ (x_1, \dots, x_n)^{d-n}, & \text{for } \frac{d-1}{d} \leq \lambda < 1. \end{cases} \quad (7)$$

Consider now an arbitrary prime p . We want to describe when $\tau(f^\lambda_p)$ is given by the same formula as the reduction of the multiplier ideals. We again distinguish two cases, according to the value of d .

Case 1. Suppose that $d \leq n$. One can show that in this case we have $\tau(f^\lambda_p) = \mathbf{F}_p[x_1, \dots, x_n]$ for every $\lambda < 1$ if and only if the morphism induced by the Frobenius

$$F : H^{n-2}(Y_p, \omega_{Y_p}) \rightarrow H^{n-2}(Y_p, \omega_{Y_p}^{\otimes p}) \quad (8)$$

is injective (recall that $\dim(Y_p) = n - 2$, hence $H^{n-2}(Y_p, \omega_{Y_p}) \simeq \mathbf{F}_p$).

Case 2. Suppose now that $d > n$. Since we are only interested in large values of p , we may assume that p does not divide d , and we fix e such that $p^e \equiv 1 \pmod{d}$. For $0 \leq r \leq d - n$, consider the morphism

$$T_r = f^{\frac{n+r}{d}(p^e-1)} F^e : H^{n-1}(\mathbf{P}_{\mathbf{F}_p}^{n-1}, \mathcal{O}(-n-r)) \rightarrow H^{n-1}(\mathbf{P}_{\mathbf{F}_p}^{n-1}, \mathcal{O}(-n-r)),$$

where we denote by F the morphism induced on the cohomology of the projective space by the Frobenius (T_r depends on the choice of e , but if we replace e by me , then T_r is replaced by T_r^m). Note that $H^{n-2}(Y_p, \mathcal{O}(d-n-r)) \subseteq H^{n-1}(\mathbf{P}_{\mathbf{F}_p}^{n-1}, \mathcal{O}(-n-r))$ consists of the elements annihilated by f . In particular, when $r = d - n$, T_r induces a map from $H^{n-2}(Y_p, \mathcal{O}_{Y_p})$ to itself, which coincides with the one induced by the e th iterate of the Frobenius on Y_p .

It is easy to see that we always have $\tau(f^\lambda_p) \subseteq (x_1, \dots, x_n)^r$ if $\lambda \in [(n+r-1)/d, (n+r)/d)$. Moreover, this is an equality for all such λ if and only if the above map T_r is injective (in fact, it is enough to check the injectivity of this map only on $H^{n-2}(Y_p, \mathcal{O}(d-n-r))$). In particular, Conjecture 3.3 for $\lambda \in [(d-1)/d, 1)$ predicts that there are infinitely many primes p such that the map induced by the Frobenius

$$H^{n-2}(Y_p, \mathcal{O}_{Y_p}) \rightarrow H^{n-2}(Y_p, \mathcal{O}_{Y_p})$$

is injective.

Remark 3.6. (See [MTW], Example 4.6.) Conjecture 3.3 holds in the case $d < n$ in the above example by a standard argument. Indeed, in that case Y is a Fano variety, and it is known that if $p \gg 0$, then Y_p is Frobenius split (see, for example, Exercise 1.6.E(4) in [BK]). Moreover, Y_p is Frobenius split if and only if the morphism (8) is injective.

On the other hand, the case $d = n$ already seems very hard. One case that is understood is when $d = n = 3$ (that is, when Y is an elliptic curve). We see that in this case we have $\tau(f_p^\lambda) = \mathbb{F}_p[x_1, x_2, x_3]$ for every $\lambda < 1$ if and only if Y_p is ordinary. The behavior when p varies depends on whether Y has complex multiplication or not. When Y has complex multiplication, then there is a quadratic field K such that if Y_p is nonsingular, then Y_p is ordinary if and only if p is completely split in K . On the other hand, if Y does not have complex multiplication, then by a result of Serre [Ser] the set of primes p for which Y_p is not ordinary has density zero (note also that Elkies [El] proved that there are infinitely many such primes). However, in this case too there is a number field K such that whenever p is completely split in K , the curve Y_p is ordinary. This follows by taking first a finite extension K' of \mathbb{Q} containing all ℓ -torsion points of Y , where ℓ is an odd prime. Then one can show that if $p \neq 2, 3, \ell$ is a prime that is completely split in K' , then Y_p is ordinary (see Exercise 5.11 in [Sil]). It is enough to take K a finite extension of K' in which 2, 3, and ℓ are not completely split.

Remark 3.7. Motivated by Example 3.4 and the above remark (see also [MTW] for other examples) one can ask whether in the context of Conjecture 3.3 one can always find a number field K such that whenever p is completely split in K , we have $\tau(f_p^\lambda) = \mathcal{J}(f^\lambda)_p$. This would give a positive answer to the conjecture by Čebotarev's density theorem. The advantage of such a statement is that, in particular, it would imply that the intersection of two such sets is again infinite: if K is a finite extension of two number fields K_1 and K_2 , then whenever p is completely split in K , it is completely split also in K_1 and K_2 .

4. The action of Euler operators in positive characteristic

From now on we work in the following setup. Let R be an F -finite regular domain of positive characteristic p . We denote by $D_R \subseteq \text{End}_{\mathbb{F}_p}(R)$ the ring of (absolute) differential operators on R . In order to avoid the possible confusion with the product in D_R , we denote the action of $P \in D_R$ on $h \in R$ by $P \bullet h$. Since R is an F -finite regular ring, D_R admits the following description (see [Bli], Proposition 3.3). For every $e \geq 0$, let $D_R^e = \text{End}_{R^{p^e}}(R)$, in particular $D_R^0 = R$. We have $D_R^e \subseteq D_R^{e+1}$ and

$$D_R = \bigcup_{e \in \mathbb{N}} D_R^e.$$

We also consider the polynomial ring $R[t]$, which is again a regular F -finite domain. The corresponding rings of differential operators will be denoted by $D_{R[t]}$ and $D_{R[t]}^e$. For every $m \geq 1$, the divided power differential operator $\partial_t^{[m]}$ acts on $R[t]$ by

$$\partial_t^{[m]} \bullet at^r = a \binom{r}{m} t^{r-m}$$

for every $a \in R$ (we follow the usual convention that $\binom{r}{m} = 0$ if $r < m$). If e is a nonnegative integer, then

$$D_{R[t]}^e = D_R^e[t, \partial_t^{[m]} \mid m < p^e]$$

(it is enough to consider only those $\partial_t^{[m]}$ with m a power of p). For $m \geq 1$, we put $\vartheta_m := \partial_t^{[m]} t^m$. In particular, $\vartheta_1 = \partial_t t$ is the Euler operator that appeared in Section 2.

We will repeatedly use the well-known theorem of Lucas (see [Luc], and also [Gra]): if we consider the p -adic decompositions of m and n , that is, $m = \sum_{i=0}^r a_i p^i$ and $n = \sum_{i=0}^r b_i p^i$, where $a_i, b_i \in \{0, \dots, p-1\}$, then

$$\binom{m}{n} \equiv \prod_{i=0}^r \binom{a_i}{b_i} \pmod{p}.$$

For future reference, we collect in the following lemma some computations in $D_{R[t]}$. They are standard and at least some of them are well known, but we include a proof for the benefit of the reader.

Lemma 4.1. *We have the following identities:*

- (i) $[t, \vartheta_m] = -\vartheta_{m-1} \cdot t$ for every $m \geq 1$ (with the convention that $\vartheta_0 = 1$).
- (ii) $[\partial_t^{[p^e]}, t^{p^e}] = 1$ for every $e \geq 0$.
- (iii) $(\partial_t^{[p^e]})^r (t^{p^e})^r = \prod_{j=0}^{r-1} (\vartheta_{p^e} + j)$.
- (iv) $\frac{(sr)!}{(s!)^r} \partial_t^{[sr]} = (\partial_t^{[s]})^r$.
- (v) For every i and j , we have $\binom{i+j}{i} \partial_t^{[i+j]} = \partial_t^{[i]} \partial_t^{[j]}$.
- (vi) For every i and j , we have $[\vartheta_i, \vartheta_j] = 0$.
- (vii) If $a_0, \dots, a_e \in \{0, \dots, p-1\}$, and $m = \sum_{i=0}^e a_i p^i$, then

$$\vartheta_m = \prod_{i=0}^e \frac{1}{a_i!} \cdot \prod_{j=0}^{a_i-1} (\vartheta_{p^i} + j),$$

where if $a_i = 0$ the product $\prod_{j=0}^{a_i-1} (\vartheta_{p^i} + j)$ is understood to be 1.

(viii) For every i and j we have

$$[\partial_t^{[p^i]}, \vartheta_p^j] = \begin{cases} \partial_t^{[p^i]}, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. In order to prove (i), it is enough to show that both sides give the same result when applied to a monomial t^n , for $n \geq 0$. Note that $\vartheta_m \bullet t^r = \binom{m+r}{r} t^r$. Therefore we have

$$[t, \vartheta_m] \bullet t^n = \left(\binom{m+n}{m} - \binom{m+n+1}{m} \right) t^{n+1} = -\binom{m+n}{m-1} t^{n+1} = -\vartheta_{m-1} t \bullet t^n.$$

We similarly deduce (ii):

$$[\partial_t^{[p^e]}, t^{p^e}] \bullet t^n = \left(\binom{n+p^e}{p^e} - \binom{n}{p^e} \right) t^n = t^n,$$

where the second equality follows from Lucas' Theorem. Moreover, (ii) implies by induction on r that

$$[\partial_t^{[p^e]}, (t^{p^e})^r] = r(t^{p^e})^{r-1}.$$

As a consequence, we easily get (iii), also by induction on r .

The formulas in (iv) and (v) follow, too, by evaluating both sides on every t^n :

$$\frac{(sr)!}{(s!)^r} \partial_t^{[sr]} \bullet t^n = \frac{(sr)!}{(s!)^r} \binom{n}{sr} t^{n-rs} = \binom{n}{s} \cdot \binom{n-s}{s} \cdots \binom{n-(r-1)s}{s} t^{n-rs} = (\partial_t^{[s]})^r \bullet t^n,$$

$$\binom{i+j}{i} \partial_t^{[i+j]} \bullet t^n = \binom{i+j}{i} \binom{n}{i+j} t^{n-(i+j)} = \binom{n-j}{i} \binom{n}{j} t^{n-(i+j)} = \partial_t^{[i]} \partial_t^{[j]} \bullet t^n.$$

To get (vi), note that

$$\vartheta_i \vartheta_j \bullet t^n = \binom{n+i}{i} \binom{n+j}{j} t^n = \vartheta_j \vartheta_i \bullet t^n.$$

We now show (vii). Note that by Lucas' Theorem, for every $0 \leq i \leq e$ and every $a \in \{1, \dots, p-1\}$ we have in \mathbf{F}_p

$$\binom{a_i p^i + \dots + a_e p^e}{a_i p^i} = 1 \quad \text{and} \quad \frac{(ap^e)!}{(p^e!)^a} = \prod_{j=1}^a \binom{jp^e}{p^e} = \prod_{j=1}^a j = a!.$$

Using this and (v), (iv), (iii), plus the fact that $[\partial_t^{[p^i]}, t^{p^j}] = 0$ whenever $i < j$, we get

$$\begin{aligned} \partial_t^{[m]} t^m &= \left(\prod_{i=0}^e \partial_t^{[a_{e-i} p^{e-i}]} \right) \cdot t^m = \prod_{i=0}^e \frac{(\partial_t^{[p^{e-i}]})^{a_{e-i}}}{a_{e-i}!} \cdot \prod_{i=0}^e t^{a_{e-i} p^{e-i}} \\ &= \prod_{i=0}^e \frac{1}{a_{e-i}!} (\partial_t^{[p^{e-i}]})^{a_{e-i}} (t^{p^{e-i}})^{a_{e-i}} = \prod_{i=0}^e \frac{1}{a_i!} \cdot \prod_{j=0}^{a_i-1} (\vartheta_{p^i} + j). \end{aligned}$$

To avoid trivial special cases, the above products can be taken to run over those i such that $a_{e-i} \neq 0$.

In order to prove (viii), we evaluate both sides on t^m . Since $\partial_t^{[q]} \bullet t^m = \binom{m}{q} t^{m-q}$ and $\vartheta_q \bullet t^m = \binom{q+m}{q} t^m$ for every m and q , we deduce that

$$[\partial_t^{[p^i]}, \vartheta_{p^j}] \bullet t^m = \binom{m}{p^i} \left(\binom{m+p^j}{p^j} - \binom{m+p^j-p^i}{p^j} \right) t^{m-p^i}.$$

If we write $m+p^j = b_0 + b_1 p + \dots$, with all $b_i \in \{0, \dots, p-1\}$, then $\binom{m+p^j}{p^j} = b_j$ in \mathbf{F}_p (this is a consequence of Lucas' Theorem). We deduce that

$$\binom{m+p^j}{p^j} - \binom{m}{p^j} \equiv 1 \pmod{p},$$

which gives our assertion when $i = j$.

Suppose now that $i \neq j$. We may assume that $p^i \leq m$, since otherwise $\binom{m}{p^i} = 0$. If $i > j$, then the coefficients of p^j in the p -adic expansions of $m+p^j$ and $m+p^j-p^i$ are the same, hence $\binom{m+p^j}{p^j} = \binom{m+p^j-p^i}{p^j}$ in \mathbf{F}_p . On the other hand, if $i < j$, then the coefficients of p^i in the p -adic expansions of m and $m+p^j$ are the same. Then either they are equal to zero, in which case $\binom{m}{p^i} = 0$, or they are positive, and then $m+p^j$ and $m+p^j-p^i$ have the same coefficient of p^j in their p -adic expansion. In either case, we get

$$\binom{m}{p^i} \left(\binom{m+p^j}{p^j} - \binom{m+p^j-p^i}{p^j} \right) \equiv 0 \pmod{p}. \quad \square$$

It is easy to deduce from Lemma 4.1 the fact that the operators $\vartheta_1, \vartheta_p, \dots, \vartheta_{p^{e-1}}$ admit a common basis of eigenvectors on every $D_{R[t]}^e$ -module.

Proposition 4.2. *If M is a $D_{R[t]}^e$ -module, then there is a unique decomposition*

$$M = \bigoplus_{i_1, \dots, i_e \in \mathbb{F}_p} M_{i_1, \dots, i_e},$$

where for $1 \leq \ell \leq e$, the operator $\vartheta_{p^{\ell-1}}$ acts on M_{i_1, \dots, i_e} by $-i_\ell$. Moreover, each M_{i_1, \dots, i_e} is a D_R^e -module, and every morphism of $D_{R[t]}^e$ -modules preserves this decomposition.

Proof. Assertion (iii) in the lemma implies that

$$\prod_{j=0}^{p-1} (\vartheta_{p^e} + j) = 0$$

for every $e \geq 0$. Indeed, it is enough to show that $(\vartheta_t^{[p^e]})^p = 0$, and this follows from (iv), since $\frac{p^{e+1}}{(p^e)!p}$ is divisible by p .

Moreover, it follows from (vi) that the ϑ_{p^e} are pairwise commuting operators. This gives the existence of the decomposition in the proposition, and the other assertions are immediate. \square

Remark 4.3. If M is a $D_{R[t]}^e$ -module, then M is in particular a $D_{R[t]}^{e-1}$ -module, hence we get a corresponding decomposition as such. It is clear that these decompositions are compatible, that is

$$M_{i_1, \dots, i_{e-1}} = \bigoplus_{j \in \mathbb{F}_p} M_{i_1, \dots, i_{e-1}, j}.$$

Remark 4.4. If M is as above, and if $m = b_1 + b_2 p + \dots + b_e p^{e-1}$, where all $b_i \in \{0, \dots, p-1\}$, then ϑ_m acts on M_{i_1, \dots, i_e} by

$$\prod_{\ell=1}^e (-1)^{b_\ell} \binom{i_\ell}{b_\ell}.$$

This is a consequence of the formula in Lemma 4.1(vii).

Proposition 4.5. *If M is a $D_{R[t]}^e$ -module, then for every $1 \leq \ell \leq e$ we have*

$$(i) \quad \vartheta_t^{[p^{\ell-1}]} \cdot M_{i_1, \dots, i_e} \subseteq M_{i_1, \dots, i_{\ell+1}, \dots, i_e},$$

$$(ii) \quad t^{p^{\ell-1}} \cdot M_{i_1, \dots, i_e} \subseteq \begin{cases} M_{i_1, \dots, i_{\ell-1}, \dots, i_e}, & \text{if } i_\ell \neq 0; \\ M_{i_1, \dots, p-1, i_{\ell+1}-1, \dots, i_e}, & \text{if } i_\ell = 0, i_{\ell+1} \neq 0; \\ \vdots & \vdots \\ M_{i_1, \dots, p-1, \dots, p-1, i_e-1}, & \text{if } i_\ell = \dots = i_{e-1} = 0, i_e \neq 0; \\ M_{i_1, \dots, i_{\ell-1}, p-1, \dots, p-1}, & \text{if } i_\ell = \dots = i_e = 0. \end{cases}$$

Proof. The first formula follows from Lemma 4.1(viii). For the second assertion, it is enough to consider the case $\ell = 1$, since the general case follows applying this one $p^{\ell-1}$ times. Note first that by Remark 4.4, for every $1 \leq e' \leq e$ the operator $\vartheta_{p^{e'}-1}$ is described on each component by

$$\vartheta_{p^{e'}-1}|_{M_{j_1, \dots, j_e}} = \begin{cases} \text{Id}_{M_{j_1, \dots, j_e}}, & \text{if } j_1 = \dots = j_{e'} = p-1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $w \in M_{i_1, \dots, i_e}$. We show by induction on $\ell \in \{1, \dots, e\}$ that

$$\vartheta_{p^{\ell-1}}(tw) = \begin{cases} -(i_\ell - 1)tw, & \text{if } i_1 = \dots = i_{\ell-1} = 0; \\ -i_\ell tw, & \text{otherwise} \end{cases}$$

(with the convention that when $\ell = 1$ we are always in the first case). This implies the assertion in (ii) for multiplication by t .

By Lemma 4.1(i), we have

$$\vartheta_{p^{\ell-1}}(tw) = t\vartheta_{p^{\ell-1}}(w) + \vartheta_{p^{\ell-1}-1}(tw) \quad (9)$$

(with the convention that $\vartheta_0 = 1$). This gives $\vartheta_1(tw) = -(i_1 - 1)tw$. Suppose now that we know the formula for $\vartheta_{p^{\ell'-1}}(tw)$ for all $\ell' \leq \ell - 1$. In particular, this implies that $tw \in M_{j_1, \dots, j_{\ell-1}}$ for some $j_1, \dots, j_{\ell-1}$, and $j_1 = \dots = j_{\ell-1} = p-1$ if and only if $i_1 = \dots = i_{\ell-1} = 0$. Our description of $\vartheta_{p^{\ell-1}-1}$ gives

$$\vartheta_{p^{\ell-1}-1}(tw) = \begin{cases} tw, & \text{if } i_1 = \dots = i_{\ell-1} = 0; \\ 0, & \text{otherwise.} \end{cases}$$

The formula for $\vartheta_{p^{\ell-1}}(tw)$ now follows from this and (9). The proof of (ii) is now complete. \square

Remark 4.6. It follows from Proposition 4.5(ii) that for every $D_{R[t]}^e$ -module M and every $i_1, \dots, i_e \in \mathbf{F}_p$ the component M_{i_1, \dots, i_e} is a $D_R^e[t^{p^e}]$ -submodule.

Example 4.7. If we write $m = \sum_{i \geq 1} a_i p^{i-1}$, with $0 \leq a_i \leq p-1$, then we have seen that $\vartheta_{p^e} \bullet t^m = \binom{m+p^e}{p^e} t^m = (a_e + 1)t^m$. It follows that if $M = R[t]$, then for every $a_1, \dots, a_e \in \{0, \dots, p-1\}$, the component $R[t]_{a_1, \dots, a_e}$ of $R[t]$ is free over $R[t^{p^e}]$ with basis t^m , where $m = \sum_{i=1}^e (p-1-a_i)p^{i-1}$.

5. The D -module B_f in positive characteristic

We now specialize the discussion in the previous section to the case of the module B_f . Suppose that $f \in R$ is nonzero. By analogy with the situation in Section 2, we put

$$B_f := R[t]_{f-t}/R[t].$$

Since $R[t]$ is naturally a $D_{R[t]}$ -module, and since every localization of a $D_{R[t]}$ -module is again a $D_{R[t]}$ -module, we see that B_f has a natural structure of $D_{R[t]}$ -module. We want to study the decomposition of B_f under the action of Euler operators.

In order to describe this decomposition we will make use of the fact that B_f is a unit F -module. We start with a lemma that applies to arbitrary unit F -modules. For the theory of unit F -modules we refer to [Lyu] or [Bli]. Let $R[t]^{(e)}$ denote the $R[t]$ -bimodule $R[t]$, with the left module structure being the usual one, and the right one being induced by the e th iterated Frobenius. A unit F -module over $R[t]$ is an $R[t]$ -module M , together with a map $F: M \rightarrow M$ that is semilinear with respect to

the Frobenius morphism on $R[t]$, and such that the induced $R[t]$ -linear map $\nu_1 : R[t]^{(1)} \otimes_{R[t]} M \rightarrow M$ given by $\nu_1(h \otimes w) = hF(w)$ is an isomorphism. Iterating, we get isomorphisms

$$\nu_e : R[t]^{(e)} \otimes_{R[t]} M \rightarrow M$$

for every $e \geq 1$. Note that $R[t]^{(e)} \otimes_{R[t]} M$ has a natural $D_{R[t]}^e$ -module structure such that $P \cdot (h \otimes w) = (P \bullet h) \otimes w$. It follows that a unit F -module M over $R[t]$ has a canonical $D_{R[t]}^e$ -module structure such that ν_e is an isomorphism of $D_{R[t]}^e$ -modules. In fact, letting e vary one gets a $D_{R[t]}$ -module structure on M .

Lemma 5.1. *For every unit F -module M over $R[t]$, and every $i_1, \dots, i_e \in \{0, \dots, p-1\}$, the component M_{i_1, \dots, i_e} is generated as an R -module by $t^m F^e(M)$, where $m = \sum_{\ell=1}^e (p - i_\ell - 1)p^{\ell-1}$.*

Proof. Since ν_e is an isomorphism of $D_{R[t]}^e$ -modules, it induces an isomorphism between the corresponding components of the two $D_{R[t]}^e$ -modules. Therefore every element in M_{i_1, \dots, i_e} can be written as $\nu_e(h \otimes w) = hF^e(w)$, for some $h \in R[t]_{i_1, \dots, i_e}$. We now deduce our assertion from Example 4.7. \square

Corollary 5.2. *If M is a unit F -module over $R[t]$, then $F(M_{i_1, \dots, i_e}) \subseteq M_{p-1, i_1, \dots, i_e}$.*

Let M be a unit F -module over $R[t]$. Given $w \in M$, for every $e \geq 1$ and every $i_1, \dots, i_e \in \{0, \dots, p-1\}$, we put

$$w_{i_1, \dots, i_e} := \nu_e(t^{\sum_{\ell=1}^e (p - i_\ell - 1)p^{\ell-1}} \otimes w) = t^{\sum_{\ell=1}^e (p - i_\ell - 1)p^{\ell-1}} F^e(w) \in M.$$

It follows from Lemma 5.1 that $w_{i_1, \dots, i_e} \in M_{i_1, \dots, i_e}$. Note that the induced map $M \rightarrow M_{i_1, \dots, i_e}$ that takes each w to w_{i_1, \dots, i_e} is semilinear with respect to the e th iterate of the Frobenius morphism on $R[t]$.

We now turn to the case of the module B_f . The $D_{R[t]}$ -module structure on B_f is induced by a unit F -module structure, such that the $R[t]$ -linear isomorphism $\nu_1 : R[t]^{(1)} \otimes_{R[t]} B_f \rightarrow B_f$ is given by $\nu_1(a \otimes u) = au^p$. Therefore the induced isomorphism ν_e satisfies $\nu_e(a \otimes u) = au^{p^e}$.

Note that B_f is a free R -module with basis $\{\delta_m\}_{m \geq 0}$, where δ_m is the class of $\frac{1}{(f-t)^{m+1}}$ in B_f . A special role is played by $\delta := \delta_0$. It follows by direct computation that for every $e \geq 0$ we have

$$t^{p^e} \cdot \delta_m = f^{p^e} \delta_m - \delta_{m-p^e} \quad (\delta_i = 0 \text{ for } i < 0), \quad (10)$$

$$\partial_t^{[p^e]} \cdot \delta_m = \binom{m+p^e}{p^e} \delta_{m+p^e}. \quad (11)$$

Suppose now that $e \geq 1$ is fixed, and consider $i_1, \dots, i_e \in \{0, \dots, p-1\}$, and a nonnegative integer m . We put

$$Q_{i_1, \dots, i_e}^m := (\delta_m)_{i_1, \dots, i_e} = \nu_e(t^{\sum_{\ell=1}^e (p - i_\ell - 1)p^{\ell-1}} \otimes \delta_m).$$

We will see that when m varies, these elements give an R -basis of $(B_f)_{i_1, \dots, i_e}$. We start by writing these elements in the basis given by the δ_i .

Lemma 5.3. *With the above notation, for every $i_1, \dots, i_e \in \{0, 1, \dots, p-1\}$ and every nonnegative integer m we have*

$$Q_{i_1, \dots, i_e}^m = (-1)^{i_1 + \dots + i_e} \sum_{j_1, \dots, j_e} \binom{i_1 + j_1}{i_1} \dots \binom{i_e + j_e}{i_e} f^{\sum_{\ell=1}^e j_\ell p^{\ell-1}} \delta_{mp^e + (i_1 + j_1) + \dots + (i_e + j_e)p^{e-1}}, \quad (12)$$

where the sum is over the integers j_1, \dots, j_e such that $0 \leq j_\ell \leq p - i_\ell - 1$ for all ℓ .

Proof. The right-hand side of (12) is equal to $v_e(h \otimes \delta_m)$, where

$$h = (-1)^{i_1 + \dots + i_e} \prod_{\ell=1}^e \left(\sum_{j_\ell=0}^{p-i_\ell-1} \binom{i_\ell + j_\ell}{i_\ell} f^{j_\ell} p^{\ell-1} (f-t)^{p^{\ell-1}(p-i_\ell-j_\ell-1)} \right).$$

Consider now

$$h_\ell := \sum_{j_\ell=0}^{p-i_\ell-1} \binom{i_\ell + j_\ell}{i_\ell} f^{j_\ell} (f-t)^{p-i_\ell-j_\ell-1}.$$

It follows from Lemma 5.4 below that we may write in the fraction field of $R[t]$

$$h_\ell = (f-t)^{p-1-i_\ell} \sum_{j_\ell=0}^{p-i_\ell-1} \binom{i_\ell + j_\ell}{i_\ell} \left(\frac{f}{f-t} \right)^{j_\ell} = (f-t)^{p-1-i_\ell} \cdot \left(1 - \frac{f}{f-t} \right)^{p-1-i_\ell} = (-t)^{p-1-i_\ell}.$$

We deduce that $h = \prod_{\ell=1}^e (t^{p-1-i_\ell})^{p^{\ell-1}}$, which implies the formula in the lemma. \square

Lemma 5.4. We have the following identity in the polynomial ring $\mathbb{F}_p[x]$

$$\sum_{j=0}^{p-i-1} \binom{i+j}{i} x^j = (1-x)^{p-i-1}$$

for every $i \in \{0, \dots, p-1\}$.

Proof. We have

$$\begin{aligned} \sum_{j=0}^{p-i-1} \binom{i+j}{i} x^j &= \frac{1}{i!} \left(\sum_{j=0}^{p-1} x^j \right)^{(i)} = \frac{1}{i!} \left(\frac{1-x^p}{1-x} \right)^{(i)} = \frac{1}{i!} ((1-x)^{p-1})^{(i)} \\ &= (-1)^i \frac{(p-1)(p-2) \cdots (p-i)}{i!} (1-x)^{p-1-i} = (1-x)^{p-1-i}. \quad \square \end{aligned}$$

We can now describe the decomposition of B_f under the action of the Euler operators.

Theorem 5.5. For every $e \geq 1$, and $i_1, \dots, i_e \in \{0, \dots, p-1\}$, the set $\{Q_{i_1, \dots, i_e}^m \mid m \geq 0\}$ gives an R -basis of $(B_f)_{i_1, \dots, i_e}$. Moreover, if $1 \leq \ell \leq e$, then the following hold:

- (i) $\partial_t^{[p^{\ell-1}]} \cdot Q_{i_1, \dots, i_e}^m = -(i_\ell + 1) Q_{i_1, \dots, i_{\ell+1}, \dots, i_e}^m$ (when $i_\ell = p-1$, this expression is understood to be zero).
- (ii) $t^{p^{\ell-1}} \cdot Q_{i_1, \dots, i_e}^m = \begin{cases} Q_{i_1, \dots, i_{\ell-1}, i_\ell-1, \dots, i_e}^m, & \text{if } i_\ell \neq 0; \\ Q_{i_1, \dots, p-1, i_{\ell+1}-1, \dots, i_e}^m, & \text{if } i_\ell = 0, i_{\ell+1} \neq 0; \\ \vdots & \vdots \\ Q_{i_1, \dots, p-1, \dots, p-1, i_e-1}^m, & \text{if } i_\ell = \dots = i_{e-1} = 0, i_e \neq 0; \\ f^{p^\ell} Q_{i_1, \dots, i_{\ell-1}, p-1, \dots, p-1}^m - Q_{i_1, \dots, i_{\ell-1}, p-1, \dots, p-1}^{m-1}, & \text{if } i_\ell = \dots = i_e = 0 \end{cases}$
- (where we put $Q_{j_1, \dots, j_e}^{-1} = 0$ for every j_1, \dots, j_e).
- (iii) $R \cdot Q_{i_1, \dots, i_e}^m$ is a D_R^e -submodule of B_f , isomorphic to R by an isomorphism that takes Q_{i_1, \dots, i_e}^m to 1.

Proof. We claim that the Q_{i_1, \dots, i_e}^m , when i_1, \dots, i_e , and m vary, give an R -basis of B_f . Indeed, we see that in Lemma 5.3, the term in (12) corresponding to $j_1 = \dots = j_e = 0$ is

$$(-1)^{i_1 + \dots + i_e} \delta_{mp^e + i_1 + \dots + i_e p^{e-1}},$$

and all the other terms are in the linear span of the $\delta_{mp^e + i'_1 + \dots + i'_e p^{e-1}}$, where $i_\ell \leq i'_\ell \leq p-1$ for all ℓ , and $i'_\ell > i_\ell$ for some ℓ . Since the δ_i with $i \geq 0$ give an R -basis of B_f , we deduce our claim. Since each Q_{i_1, \dots, i_e}^m lies in $(B_f)_{i_1, \dots, i_e}$, we get the first assertion in the theorem.

If $P \in D_{R[t]}^e$, we may compute $P \cdot Q_{i_1, \dots, i_e}^m$ as $v_e(P \bullet t^{\sum_{\ell=1}^e (p-i_\ell-1)p^{\ell-1}} \otimes \delta_m)$. If $P \in D_R^e$ and $h \in \mathbb{F}_p[t] \subseteq R[t]$, then $P \bullet h = h(P \bullet 1)$. Therefore $P \cdot Q_{i_1, \dots, i_e}^m = (P \bullet 1)Q_{i_1, \dots, i_e}^m$, which implies (iii).

Note that if $1 \leq \ell \leq e$, and if we write $n = i_1 + \dots + i_e p^{e-1} + mp^e$, with $i_1, \dots, i_e \in \{0, \dots, p-1\}$ and $m \geq 0$, then

$$\partial_t^{[p^{\ell-1}]} \bullet t^n = \binom{n}{p^{\ell-1}} t^{n-p^{\ell-1}} = i_\ell t^{n-p^{\ell-1}}$$

(the second equality follows from Lucas' Theorem). If we take $n = \sum_{\ell=1}^e (p-i_{\ell'}-1)p^{\ell'-1}$, then we get

$$\partial_t^{[p^{\ell-1}]} \cdot Q_{i_1, \dots, i_e}^m = v_e((\partial_t^{[p^{\ell-1}]} \bullet t^n) \otimes \delta_m) = (p-i_\ell-1)v_e(t^{n-p^{\ell-1}} \otimes \delta_m) = -(i_\ell+1)Q_{i_1, \dots, i_{\ell}+1, \dots, i_e},$$

hence (i). We also have

$$t^{p^{\ell-1}} \cdot Q_{i_1, \dots, i_e}^m = v_e(t^{(p-i_1-1)+\dots+(p-(i_\ell-1)+1)p^{\ell-1}+\dots+(p-i_e-1)p^{e-1}} \otimes \delta_m).$$

The formula in (ii) is an immediate consequence. \square

Remark 5.6. It follows from the formula in Lemma 5.3 that the Q_{i_1, \dots, i_e}^m with $0 \leq i_\ell \leq p-1$ for all ℓ , and with $m \leq m_0$, give an R -basis of the $D_{R[t]}^e$ -submodule $\bigoplus_{i \leq (m_0+1)p^{e-1}} R \cdot \delta_i$.

Remark 5.7. It would be useful to have an explicit formula for the change of basis when we replace e by $e+1$. In the case $m=0$ we have the following formula:

$$Q_{i_1, \dots, i_e}^0 = \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} f^{jp^e} Q_{i_1, \dots, i_e, j}^0 \quad (13)$$

for every $i_1, \dots, i_e \in \{0, 1, \dots, p-1\}$. Indeed, we have

$$\begin{aligned} Q_{i_1, \dots, i_e}^0 &= t^{\sum_{\ell=1}^e (p-i_\ell-1)p^{\ell-1}} \cdot \frac{1}{(f-t)^{p^e}} = t^{\sum_{\ell=1}^e (p-i_\ell-1)p^{\ell-1}} (f-t)^{p^e(p-1)} \cdot \frac{1}{(f-t)^{p^{e+1}}} \\ &= \sum_{j=0}^{p-1} (-1)^{p-1-j} \binom{p-1}{j} f^{jp^e} t^{p^e(p-j-1)+\sum_{\ell=1}^e (p-i_\ell-1)p^{\ell-1}} \cdot \frac{1}{(f-t)^{p^{e+1}}} \\ &= \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} f^{jp^e} Q_{i_1, \dots, i_e, j}^0. \end{aligned}$$

6. Bernstein–Sato polynomials in positive characteristic

We keep the notation in the previous section. Motivated by the analogy with the situation described in Section 2, we study some modules over rings of differential operators of R . For every positive integer e , consider the D_R^e -submodule of B_f

$$M_f^e := D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^e-1}] \cdot \delta.$$

The union of all M_f^e is the D_R -module

$$M_f := \varinjlim_e M_f^e = D_R[\vartheta_{p^i} \mid i \geq 0] \cdot \delta.$$

We use the decomposition in Theorem 5.5 to give an explicit description of M_f^e .

Proposition 6.1. *With the above notation, we have*

$$M_f^e = \bigoplus_{i_1, \dots, i_e=0}^{p-1} (D_R^e \bullet f^{i_1+i_2p+\dots+i_ep^{e-1}}) Q_{i_1, \dots, i_e}^0. \quad (14)$$

Proof. We first show that

$$\delta = \sum_{i_1, \dots, i_e=0}^{p-1} (-1)^{i_1+\dots+i_e} \binom{p^e-1}{i_1+i_2p+\dots+i_ep^{e-1}} f^{i_1+i_2p+\dots+i_ep^{e-1}} Q_{i_1, \dots, i_e}^0. \quad (15)$$

To see this, note that

$$\begin{aligned} \delta &= v_e((f-t)^{p^e-1} \otimes \delta) \\ &= v_e \left(\sum_{i_1, \dots, i_e=0}^{p-1} \binom{p^e-1}{i_1+i_2p+\dots+i_ep^{e-1}} f^{i_1+i_2p+\dots+i_ep^{e-1}} \cdot (-t)^{(p-1-i_1)+\dots+p^{e-1}(p-1-i_e)} \otimes \delta \right) \\ &= \sum_{i_1, \dots, i_e=0}^{p-1} (-1)^{i_1+\dots+i_e} \binom{p^e-1}{i_1+i_2p+\dots+i_ep^{e-1}} f^{i_1+\dots+i_ep^{e-1}} Q_{i_1, \dots, i_e}^0. \end{aligned}$$

Note now that the binomial coefficients in (15) are all different from zero. Indeed, it follows from Lucas' Theorem that

$$\binom{p^e-1}{i_1+i_2p+\dots+i_ep^{e-1}} \equiv \prod_{\ell=1}^e \binom{p-1}{i_\ell} \pmod{p}.$$

By Theorem 5.5, each $R \cdot Q_{i_1, \dots, i_e}^0$ is an eigenspace of $\vartheta_{p^{\ell-1}}$ with eigenvalue $-i_\ell$, and therefore M_f^e is the direct sum of its intersections with the $R \cdot Q_{i_1, \dots, i_e}^0$. Since we have an isomorphism of D_R^e -modules $R \simeq R \cdot Q_{i_1, \dots, i_e}^0$ that takes 1 to Q_{i_1, \dots, i_e}^0 , we get the decomposition (14). \square

Remark 6.2. It is easy to show that the D_R^e -submodules of R are precisely the ideals of the form $J^{[p^e]}$, for some ideal J of R (see, for example, Lemma 2.2 in [BMSm1]). Using the notation in Section 3, we see that for every $g \in R$

$$D_R^e \bullet g = (g^{[1/p^e]})^{[p^e]}.$$

Remark 6.3. The subring $D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^{e-1}}]$ of $D_{R[t]}^e$ contains all ϑ_m with $m < p^e$. This is an immediate consequence of the formula in Lemma 4.1(vii).

We are interested in the action of the operators $\vartheta_1, \vartheta_p, \dots, \vartheta_{p^{e-1}}$ on the quotient M_f^e/tM_f^e . The following lemma shows that indeed, $tM_f^e \subseteq M_f^e$, and the above operators have an induced action on the quotient module.

Lemma 6.4. For every $e \geq 1$ we have

$$tM_f^e = D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^{e-1}}] \cdot f\delta \subseteq M_f^e.$$

Proof. It is clear that we have $D_R^e[\vartheta_1, \dots, \vartheta_{p^{e-1}}] \cdot f\delta \subseteq M_f^e$. Note also that $t\delta = f\delta$, hence it is enough to prove the equality in $D_{R[t]}^e$

$$t \cdot D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^{e-1}}] = D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^{e-1}}] \cdot t.$$

Lemma 4.1(i) and Remark 6.3 give

$$[t, \vartheta_{p^i}] = -\vartheta_{p^{i-1}} \cdot t \in D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^{i-1}}]t$$

for every $i \leq e$. Since t commutes with the operators in D_R^e , we deduce by induction on $i \leq e-1$ that

$$t \cdot D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^i}] \subseteq D_R^e[\vartheta_1, \vartheta_p, \dots, \vartheta_{p^i}] \cdot t.$$

The reverse inclusion follows similarly, using the fact that for every m we have $[t, \vartheta_m] = -t \cdot \sum_{j=0}^{m-1} \vartheta_j$ (recall that $\vartheta_0 = 1$). This assertion follows in turn from Lemma 4.1(i), by induction on m . \square

Corollary 6.5. For every positive integer e we have a decomposition

$$M_f^e/tM_f^e = \bigoplus_{i_1, \dots, i_e} W_{i_1, \dots, i_e},$$

such that for every $1 \leq \ell \leq e$, the operator $\vartheta_{p^{\ell-1}}$ acts on W_{i_1, \dots, i_e} by $-i_\ell$, and

$$W_{i_1, \dots, i_e} \simeq (D_R^e \bullet f^{i_1+i_2p+\dots+i_ep^{e-1}})/(D_R^e \bullet f^{1+i_1+i_2p+\dots+i_ep^{e-1}})$$

(the i_1, \dots, i_e vary over $\{0, \dots, p-1\}$).

Proof. The assertion follows from Proposition 6.1 and Theorem 5.5(ii) and (iii). \square

Notation 6.6. Let $\Gamma_f^e \subseteq \{0, 1, \dots, p-1\}^e$ be the set of those (i_1, \dots, i_e) such that $W_{i_1, \dots, i_e} \neq \emptyset$. In other words, $(i_1, \dots, i_e) \in \Gamma_f^e$ if and only if there is a nonzero element $u \in M_f^e/tM_f^e$ such that $(\vartheta_{p^{\ell-1}} + i_\ell)u = 0$ for $1 \leq \ell \leq e$.

By analogy with the characteristic zero case, we define the Bernstein–Sato polynomial of f to be the minimal polynomial of $-\vartheta_1$ on the D_R^1 -module M_f^1/tM_f^1 . In other words, we have

$$b_f(s) = \prod_{i \in \Gamma_f^1} (s - i) \in \mathbf{F}_p[s].$$

Note that unlike in characteristic zero, this polynomial always has distinct roots.

In order to also keep track of the action of the higher ϑ_{p^e} , we introduce the *higher Bernstein–Sato polynomials* $b_f^{(e)}(s) \in \mathbf{Q}[s]$, defined by

$$b_f^{(e)}(s) = \prod_{(i_1, \dots, i_e) \in \Gamma_f^e} \left(s - \left(\frac{i_e}{p} + \dots + \frac{i_1}{p^e} \right) \right).$$

Note that $b_f \in \mathbf{F}_p[s]$ while $b_f^{(1)} \in \mathbf{Q}[s]$, but they contain the same amount of information. It follows from definition that $b_f^{(e)}$ has distinct roots, all of them in $\frac{1}{p^e} \mathbf{Z} \cap [0, 1)$. Our next goal is to relate the roots of $b_f^{(e)}$ to the F -jumping exponents of f .

Theorem 6.7. *For every $e \geq 1$, the roots of the Bernstein–Sato polynomial $b_f^{(e)}(s)$ are simple, and they are given by the rational numbers $\frac{[p^e \lambda] - 1}{p^e}$, where λ varies over the F -jumping exponents of f in $(0, 1]$.*

Before giving the proof of the theorem, we introduce some notation. Given $\lambda \in (0, 1]$, we can write it uniquely as

$$\lambda = \sum_{i \geq 1} \frac{c_i(\lambda)}{p^i}, \quad (16)$$

with all $c_i(\lambda) \in \{0, 1, \dots, p-1\}$, and such that infinitely many of the $c_i(\lambda)$ are nonzero. Note that the $c_i(\lambda)$ are determined recursively by $c_1(\lambda) = \lceil \lambda p \rceil - 1$ and $c_i(\lambda) = c_{i-1}(p\lambda - c_1(\lambda))$ for $i \geq 2$. Moreover, for every e we have

$$\frac{c_1(\lambda)}{p} + \dots + \frac{c_e(\lambda)}{p^e} = \frac{[p^e \lambda] - 1}{p^e}. \quad (17)$$

Lemma 6.8. *For every positive integer e ,*

$$\Gamma_f^e = \{(c_e(\lambda), \dots, c_1(\lambda)) \mid \lambda \in (0, 1] \text{ is an } F\text{-jumping exponent for } f\}.$$

Proof. It follows from Corollary 6.5 that $(i_1, \dots, i_e) \in \{0, 1, \dots, p-1\}^e$ lies in Γ_f^e if and only if $D_R^e \bullet f^{i_1+i_2p+\dots+i_ep^{e-1}} \neq D_R^e \bullet f^{1+i_1+i_2p+\dots+i_ep^{e-1}}$. On the other hand, for every nonnegative integer m we have

$$D_R^e \bullet f^m = ((f^m)^{[1/p^e]})^{[p^e]} = \tau(f^{m/p^e})^{[p^e]}$$

(for the second equality see, for example, Lemma 2.1 in [BMSm1]).

Since the Frobenius morphism on R is flat, for every two ideals I_1 and I_2 in R we have $I_1^{[p^e]} \subseteq I_2^{[p^e]}$ if and only if $I_1 \subseteq I_2$. Therefore $(i_1, \dots, i_e) \in \Gamma_f^e$ if and only if there is an F -jumping exponent λ of f in the interval $(\frac{m}{p^e}, \frac{m+1}{p^e}]$, where $m = i_1 + i_2p + \dots + i_ep^{e-1}$. On the other hand, it follows from the

definition of the $c_j(\lambda)$ that this is the case if and only if $m = c_1(\lambda)p^{e-1} + \dots + c_{e-1}(\lambda)p + c_e(\lambda)$. Using the fact that $i_e, c_j(\lambda) \in \{0, 1, \dots, p-1\}$, it follows that this is further equivalent with $(i_1, \dots, i_e) = (c_e(\lambda), \dots, c_1(\lambda))$, which completes the proof of the lemma. \square

Proof of Theorem 6.7. The fact that the roots of $b_f^{(e)}$ are simple is a consequence of the definition. Lemma 6.8 implies that these roots correspond to the rational numbers of the form $\frac{c_1(\lambda)}{p} + \dots + \frac{c_e(\lambda)}{p^e}$, where λ varies over the F -jumping exponents of f . Formula (17) implies the statement of the theorem. \square

Remark 6.9. It follows from Theorem 1.1 in [BMSm1] that there are finitely many (say r) F -jumping exponents of f in $(0, 1]$. Theorem 6.7 implies that the number of roots of $b_f^{(e)}$ is bounded above by r for every e , with equality for $e \gg 0$.

Remark 6.10. We can use the interpretation of the F -jumping exponents as F -thresholds (see Remark 3.1) to reinterpret Theorem 6.7 as follows. Let J be a proper ideal of R containing f (this is equivalent with $c^J(f) \leq 1$). For a given $e \geq 1$, the ratio $\frac{v^J(p^e)}{p^e}$ is a root of $b_f^{(e)}$, and all roots of $b_f^{(e)}$ are of this form (for some ideal J).

Example 6.11. If f is not invertible, then 1 is an F -jumping exponent for f : use Remark 3.1 and the fact that $c^{(f)}(f) = 1$. Since $c_i(1) = p-1$ for every i , we see that $(p-1, \dots, p-1) \in \Gamma_e$ for every $e \geq 1$. Therefore $1 - \frac{1}{p^e}$ is always a root of $b_f^{(e)}$.

Remark 6.12. It follows from Lemma 6.8 that we have a surjective map $\Gamma_f^{e+1} \rightarrow \Gamma_f^e$ that takes (i_1, \dots, i_{e+1}) to (i_2, \dots, i_{e+1}) . Note that we have another map $\Gamma_f^{e+1} \rightarrow \Gamma_f^e$, taking (i_1, \dots, i_{e+1}) to (i_1, \dots, i_e) . Indeed, by the same lemma, it is enough to show that for every F -jumping coefficient $\lambda \in (0, 1]$ for f , we have $(c_{e+1}(\lambda), \dots, c_2(\lambda)) \in \Gamma_f^e$.

It is known that if λ is an F -jumping exponent of f , then the fractional part $\{p\lambda\}$ of $p\lambda$ is also an F -jumping exponent (see Proposition 3.4 in [BMSm2]). If $p\lambda$ is not an integer, then $c_i(\{p\lambda\}) = c_{i+1}(\lambda)$ for $i \geq 1$, hence $(c_{e+1}(\lambda), \dots, c_2(\lambda)) \in \Gamma_f^e$. On the other hand, if $p\lambda = m \in \mathbf{Z}$, then $c_1(\lambda) = m-1$, and $c_i(\lambda) = p-1$ for $i \geq 2$. In this case, we get $(c_{e+1}(\lambda), \dots, c_2(\lambda)) \in \Gamma_f^e$ by Example 6.11.

Remark 6.13. Note that we have canonical maps $\varphi_e: M_f^e/tM_f^e \rightarrow M_f^{e+1}/tM_f^{e+1}$. If we denote by Q'_{i_1, \dots, i_e} the class of $f^{i_1+i_2p+\dots+i_ep^{e-1}}$ Q_{i_1, \dots, i_e}^0 in M_f^e/tM_f^e , then it follows from Remark 5.7 that

$$\varphi_e(Q'_{i_1, \dots, i_e}) = \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} Q'_{i_1, \dots, i_e, j}.$$

We will see in Example 6.15 below that it can happen that no map φ_e is injective, and that we miss a lot of information if instead of considering all M_f^e we consider only M_f .

Example 6.14. Consider the case of the cusp $f_p = x^2 + y^3 \in \mathbf{F}_p[x, y]$, with $p > 3$. We have seen in Example 3.4 that the only jumping numbers of f_p in $(0, 1]$ are c_p and 1, where $c_p = \frac{5}{6}$ if $p \equiv 1 \pmod{3}$, and $c_p = \frac{5}{6} - \frac{1}{6p}$ if $p \equiv 2 \pmod{3}$. Note that we have $1 = \sum_{e \geq 1} (p-1) \cdot \frac{1}{p^e}$ and

$$c_p = \begin{cases} \sum_{i \geq 1} \frac{5(p-1)}{6} \cdot \frac{1}{p^i}, & \text{if } p \equiv 1 \pmod{3}; \\ \frac{5p-7}{6} \cdot \frac{1}{p} + \sum_{i \geq 2} (p-1) \cdot \frac{1}{p^i}, & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

It follows from Lemma 6.8 that if $p \equiv 1 \pmod{3}$, then

$$\Gamma_{f_p}^e = \left\{ (p-1, \dots, p-1), \left(\frac{5(p-1)}{6}, \dots, \frac{5(p-1)}{6} \right) \right\}$$

for every $e \geq 1$, and if $p \equiv 2 \pmod{3}$, then

$$\Gamma_{f_p}^1 = \left\{ p-1, \frac{5p-7}{6} \right\}, \quad \Gamma_{f_p}^e = \left\{ (p-1, \dots, p-1), \left(p-1, \dots, p-1, \frac{5p-7}{6} \right) \right\} \quad \text{for } e \geq 2.$$

We deduce the formula for the Bernstein–Sato polynomial

$$b_f^{(e)}(s) = \begin{cases} (s - (1 - \frac{1}{p^e}))(s - \frac{5}{6}(1 - \frac{1}{p^e})), & \text{if } p \equiv 1 \pmod{3}; \\ (s - (1 - \frac{1}{p^e}))(s - (\frac{5p-1}{6p} - \frac{1}{p^e})), & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

In particular, we see that in $\mathbb{F}_p[s]$ we have

$$b_f(s) = \begin{cases} (s - (p-1))(s - \frac{5}{6}(p-1)) = (s+1)(s + \frac{5}{6}), & \text{if } p \equiv 1 \pmod{3}; \\ (s - (p-1))(s - \frac{5p-7}{6}) = (s+1)(s + \frac{7}{6}), & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Example 6.15. Let $f_p = x^2 + y^3 \in \mathbb{F}_p[x, y]$, where $p > 3$ is a prime with $p \equiv 2 \pmod{3}$. Using the notation in Remark 6.13, the computation in the previous example shows that for every $e \geq 2$ we have

$$M_{f_p}^e / tM_{f_p}^e = D_R^e \cdot Q'_{p-1, \dots, p-1, p-1} \oplus D_R^e \cdot Q'_{p-1, \dots, p-1, \frac{5p-7}{6}},$$

and both components are nonzero. We have

$$h_e(Q'_{p-1, \dots, p-1}) = Q'_{p-1, \dots, p-1, p-1} + (-1)^{(5p-7)/6} \binom{p-1}{(5p-7)/6} Q'_{p-1, \dots, p-1, \frac{5p-7}{6}}$$

and $h_e(Q'_{p-1, \dots, p-1, \frac{5p-7}{6}}) = 0$. In particular, the images of all $Q'_{p-1, \dots, p-1}$ in M_{f_p}/tM_{f_p} coincide, and this element generates M_{f_p}/tM_{f_p} over D_R . We deduce that all operators ϑ_{p^e} (for $e \geq 0$) are equal to the identity on M_{f_p}/tM_{f_p} .

Example 6.16. Let $f = x_1^2 + \dots + x_n^2 \in \mathbb{F}_p[x_1, \dots, x_n]$, where $p > 2$ and $n \geq 2$. It follows from Example 4.1 in [MTW] that the only F -jumping exponent of f in $(0, 1]$ is 1. Therefore $b_f^{(e)} = (s - (1 - \frac{1}{p^e}))$ for every $e \geq 1$. In particular, we have $b_f(s) = (s+1)$. Note however that if $\tilde{f} = x_1^2 + \dots + x_n^2 \in \mathbb{Q}[x_1, \dots, x_n]$, then $b_{\tilde{f}}(s) = (s+1)(s + \frac{n}{2})$, see [Kas1], Example 6.19.

Example 6.17. Let $f \in R = k[x_1, \dots, x_n]$, with k an F -finite field of characteristic $p > 0$, and suppose that there are integers d and w_1, \dots, w_n such that for every monomial $x^u = x_1^{u_1} \dots x_n^{u_n}$ with nonzero coefficient in f , we have $\sum_i u_i w_i \equiv d \pmod{p}$. We assume that $d \not\equiv 0 \pmod{p}$, hence we can write $f = \frac{1}{d} \cdot \sum_{i=1}^n w_i x_i \frac{\partial f}{\partial x_i}$. Therefore f has isolated singularities if and only if $\dim_k(R/J_f) < \infty$, where $J_f = (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$. If this is the case, then for every root $\beta \neq -1$ of b_f there is a monomial $x^u \notin J_f$ such that $\beta = -\frac{\sum_i w_i(u_i+1)}{d}$.

The argument is similar to the corresponding one in characteristic zero (see Section 6.4 in [Kas1]). It is clear that we have an isomorphism

$$M_f/tM_f \simeq D_R^1[\partial_t t]/J,$$

where $J = \{P \in D_R^1[\partial_t t] \mid P \cdot \delta \in tM_f\}$. If we put $T_f := (1 - \partial_t t) \cdot M_f/tM_f$, then $b_f(s)/(s+1)$ is the minimal polynomial of $-\partial_t t$ on T_f . Moreover, we have $T_f \simeq D_R^1[\partial_t t]/J'$, where $J' = \{Q \in D_R[\partial_t t] \mid (1 - \partial_t t)Q \in J\}$.

Let $\xi = \sum_i w_i x_i \partial_i$, where we put $\partial_i := \partial_{x_i}$. It follows by direct computation that $(\xi + d\partial_t) \cdot \delta = 0$, hence $\partial_t t + \frac{1}{d}\xi \in J$. Moreover, since $f \in J$ and

$$\left(\partial_i f + \frac{\partial f}{\partial x_i} (\partial_t t - 1) \right) \cdot \delta = 0,$$

we conclude that $\partial f/\partial x_i \in J'$ for every i . Hence we have a surjection of k -vector spaces

$$D_R^1[\partial_t t]/D_R^1\left(\partial_t t + \frac{1}{d}\xi, \partial f/\partial x_1, \dots, \partial f/\partial x_n\right) \simeq k[\partial_1, \dots, \partial_n]/(\partial_1^p, \dots, \partial_n^p) \otimes_k R/J_f \rightarrow T_f.$$

In order to describe the action of $\partial_t t$ on the left-hand side, note first that this commutes with the operators ∂_i . Furthermore, we have in this quotient module

$$(-\partial_t t) \cdot x^u = \frac{1}{d} x^u \xi = \frac{1}{d} \left(\sum_i w_i \partial_i x_i \right) x^u - \frac{1}{d} \sum_i (u_i + 1) w_i x^u,$$

and therefore

$$(-\partial_t t) \cdot (1 \otimes x^u) + \frac{\sum_i (u_i + 1) w_i}{d} (1 \otimes x^u) \in \sum_j \partial_j \cdot k[\partial_1, \dots, \partial_n]/(\partial_1^p, \dots, \partial_n^p) \otimes_k R/J_f.$$

It follows that if we consider on $k[\partial_1, \dots, \partial_n]/(\partial_1^p, \dots, \partial_n^p) \otimes_k R/J_f$ the decreasing filtration by the vector subspaces $\{W^\ell \otimes_k R/J_f\}_\ell$, where $W^\ell = (\partial_1^\ell, \dots, \partial_n^\ell)/(\partial_1^p, \dots, \partial_n^p)$, then for every $g \otimes x^u \in W^\ell \otimes_k R/J_f$ we have

$$(-\partial_t t) \cdot (g \otimes x^u) + \frac{\sum_i (u_i + 1) w_i}{d} (g \otimes x^u) \in W^{\ell+1} \otimes_k R/J_f.$$

This implies that every eigenvalue of $-\partial_t t$ on T_f is of the form $-\frac{\sum_i (u_i + 1) w_i}{d}$, for some monomial $x^u \in R \setminus J_f$.

We end by raising some questions related to the setup considered in this paper.

Question 6.18. The discreteness of the set of F -jumping exponents of f is equivalent with the fact that there is some r such that $\# \Gamma_f^e \leq r$ for every e . The rationality of these exponents is a direct consequence of their discreteness (see Theorem 3.1 in [BMSm2]). On the other hand, discreteness plus rationality implies the eventual periodicity of the components of the elements of the sets Γ_f^e , when e varies. It is possible to make a stronger periodicity statement for the modules M_f^e ?

Question 6.19. In characteristic zero, the main application of the Bernstein–Sato polynomial in the setting that we discussed is the construction of the V -filtration. Is there an analogue of the V -filtration in positive characteristic? A related question is the following: suppose that $\tilde{f} \in \mathbf{Z}[x_1, \dots, x_n]$. Is it possible to lift the V -filtration of \tilde{f} to a filtration on $\mathbf{Z}[x_1, \dots, x_n, t]_{f-y}/\mathbf{Z}[x_1, \dots, x_n, t]$? If this is the case, what can be said about the reduction modulo p of this filtration, for $p \gg 0$? Note that a minimum requirement for the V -filtration over \mathbf{Z} would be “to put the operator $\partial_t t$ in upper-triangular form.” More optimistically, one can ask about the existence of a structure that would deal at the same time with all operators $\partial_t^{[m]} t^m$, with $m \geq 1$.

Question 6.20. As in characteristic zero, one can consider the Bernstein–Sato polynomial of f with respect to an arbitrary element $w \in B_f$. These invariants seem to be particularly relevant when $w = h\delta$, for some $h \in R$. In this case they contain the same amount of information as the sets

$$\Gamma_{f,w}^e := \{(i_1, \dots, i_e) \in \{0, \dots, p-1\}^e \mid D_R^e \bullet h f^{i_1+i_2p+\dots+i_ep^{e-1}} \neq D_R^e \bullet h f^{1+i_1+i_2p+\dots+i_ep^{e-1}}\}.$$

For example, a natural question is whether the numbers $\#\Gamma_{f,w}^e$ are all bounded above by some r . Moreover, are these numbers eventually constant?

In characteristic zero, the construction of the V -filtration is based on the existence of b_f and on the rationality of its roots. On the other hand, once the existence of the V -filtration is known, then the existence of all $b_{f,w}$, and the rationality of their roots follow. Is it possible, in positive characteristic, to use the eventual periodicity of the components of the elements of the elements of the sets Γ_f^e , to prove a similar result about the sets $\Gamma_{f,w}^e$?

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